Fermat's Principle

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B.Sc Part-II
Paper-III

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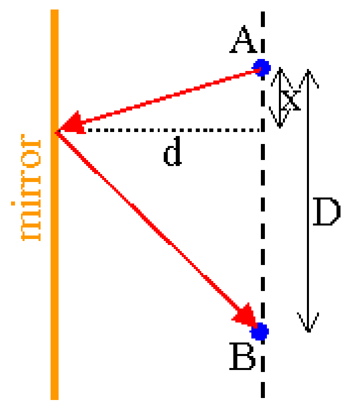
Fermat's principle

In 1650, Fermat discovered a way to explain reflection and refraction as the consequence of one single principle. It is called the principle of least time or Fermat's principle.

Assume we want light to get from point A to point B, subject to some boundary condition. For example, we want the light to bounce off a mirror or to pass through a piece of glass on its way from A to B. Fermat's principle states that of all the possible paths the light might take, that satisfy those boundary conditions, light takes the path which requires the shortest time.

(A more accurate statement of Fermat's principle: Any hypothetical small change in the actual path of a light ray would only result in a second order change in the optical path length. The first order change in the optical path length would be zero.)



Consider the diagram on the right. We want light to leave point A, bounce off the mirror, and get to point B. Let the perpendicular distance from the mirror of both A and B be d and the shortest distance between the points be D. Assume that light takes the path shown. The length of this path is

$$L = (x^2 + d^2)^{1/2} + ((D - x)^2 + d^2)^{1/2}.$$

Since the speed of light is the same everywhere along all possible paths, the shortest path requires the shortest time. To find the shortest path, we differentiate L with respect to x and set the result equal to zero. (This yields an extremum in the function L(x).)

$$\frac{dL}{dx} = \frac{x}{\sqrt{x^2 + d^2}} - \frac{(D - x)}{\sqrt{(D - x)^2 + d^2}} = 0.$$

$$\frac{x^2}{x^2 + d^2} = \frac{D^2 + x^2 - 2Dx}{D^2 + x^2 - 2Dx + d^2}$$

$$x^2\left(D^2 + x^2 - 2Dx + d^2\right) = \left(D^2 + x^2 - 2Dx\right)\left(x^2 + d^2\right)$$

After canceling equal terms on both sides we are left with

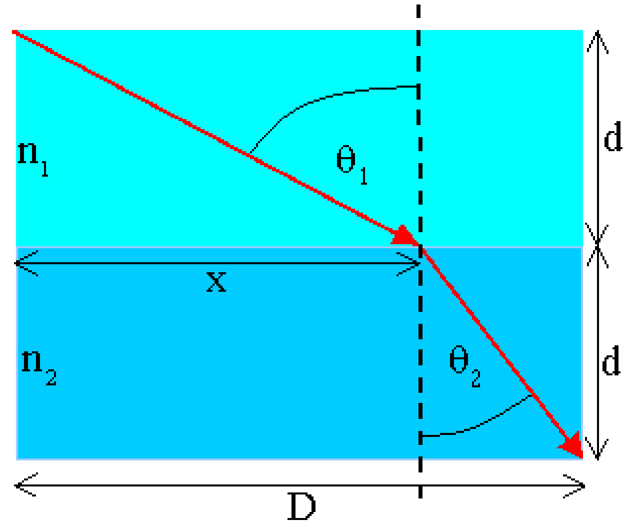
$$d^2D^2 = 2Dx, \quad \text{or } x = D/2.$$

The path that takes the shortest time is the one for which $x = D/2$, or equivalently, the one for which $\theta_i = \theta_r$. Fermat's principle yields the law of reflection.

Now assume we want light to propagate from point A to point B across the boundary between medium 1 and medium 2.

For the path shown in the figure on the right the time required is

$$t = \frac{\sqrt{x^2 + d^2}}{(c/n_1)} - \frac{\sqrt{(D-x)^2 + d^2}}{(c/n_2)}.$$



Setting dt/dx = 0 we obtain

$$n_1 \frac{x}{\sqrt{x^2 + d^2}} = n_2 \frac{D-x}{\sqrt{(D-x)^2 + d^2}},$$

or

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

Fermat's principle yields Snell's law.

In Fermat's Principle the optical path length between points A and B can be calculated in either direction, A to B or B to A, the result is the same. This leads to the principle of geometrical reversibility.

Principle of Ray Reversibility: Any actual ray of light in an optical system, if reversed in direction, will retrace the same path backward.

Fermat's Principle

This principle states that the path actually is traversed by a ray of light in passing from one point to another point is the path of least time. i.e. time

$$t = minimum$$

$$\delta t = 0$$

Explaination
Let a ray of light moves with velocity v in a medium of refractive index n and a distance ds is traversed by it in time dt. So velocity of light in that medium

$$v = \frac{ds}{dt}$$

$$dt = \frac{ds}{v}$$

Again, we have from definition of refractive index

$$n = \frac{c}{v}$$

$$\frac{1}{v} = \frac{n}{c}$$

So we get

$$dt = \frac{nds}{c}$$

$$t = \frac{1}{c}\int_A^B nds$$

$$t = \frac{1}{c}\int_A^B dL$$

where $dL = nds$ is known as optical path. so we get

$$t = \frac{L}{c}$$

We know that velocity of light is large. i.e. c is maximum velocity. so

$$t = minimum$$

$$\delta t = 0$$

So, we obtain

$$\delta L = 0$$

$$L = extremum$$

Hence, we can say that the path taken by a ray of light in passing from one point to the other is the path of minimum or maximum.

Q. With the help of Fermat's principle derive the law of reflection.

Let a ray of light AO be incident at the point O on a plane mirror MM' and after reflection it goes along OB. Here angle of incidence AON=i and angle of reflection BON=r. According to the Fig.

$CO = x$

$DO = d - x$

$CD = d$

$AC = a$

$BD = b$

$AO = \sqrt{a^2 + x^2}$

$OB = \sqrt{b^2 + (d - x)^2}$

So, optical path

$$L_{op} = nAO + nOB$$

Here, n is the refractive index of the medium.

$$L_{op} = n\sqrt{a^2 + x^2} + n\sqrt{b^2 + (d - x)^2}$$

Now, time taken by light

$$t = \frac{L}{c}$$

$$t = \frac{n}{c}\left(\sqrt{a^2 + x^2} + \sqrt{b^2 + (d - x)^2}\right)$$

According to Fermat's principle

$$\delta t = 0$$

$$\frac{n}{c}\left(\frac{x}{\sqrt{a^2+x^2}} - \frac{d-x}{\sqrt{b^2+(d-x)^2}}\right) = 0$$

Here, $\frac{n}{c} \neq 0$. So we get

$$\frac{x}{\sqrt{a^2+x^2}} - \frac{d-x}{\sqrt{b^2+(d-x)^2}} = 0$$

$$\frac{x}{\sqrt{a^2+x^2}} = \frac{d-x}{\sqrt{b^2+(d-x)^2}}$$

$$sini = sinr$$

where $sini = \frac{x}{\sqrt{a^2+x^2}}$ and $sinr = \frac{d-x}{\sqrt{b^2+(d-x)^2}}$

$$i = r$$

which is law of reflection.

Q. With the help of Fermat's principle derive the law of refraction.

Let a ray of light AO be incident at the point O on a plane mirror MM' and after refraction it goes along OB. Here angle of incidence AON=i and angle of reflection BON'=r. According to the Fig.

$CO = x$

$DO = d - x$

$CD = d$

$AC = a$

$BD = b$

$AO = \sqrt{a^2+x^2}$

$OB = \sqrt{b^2+(d-x)^2}$

So, optical path

$$L_{op} = n_1 AO + n_2 OB$$

Here, n is the refractive index of the medium.

$$L_{op} = n_1\sqrt{a^2+x^2} + n_2\sqrt{b^2+(d-x)^2}$$

Now, time taken by light

$$t = \frac{L}{c}$$

$$t = \frac{1}{c}\left(n_1\sqrt{a^2+x^2} + n_2\sqrt{b^2+(d-x)^2}\right)$$

According to Fermat's principle

$$\delta t = 0$$

$$\frac{1}{c}\left(n_1\frac{x}{\sqrt{a^2+x^2}} - n_2\frac{d-x}{\sqrt{b^2+(d-x)^2}}\right) = 0$$

Here, $\frac{1}{c} \neq 0$. So we get

$$n_1\frac{x}{\sqrt{a^2+x^2}} - n_2\frac{d-x}{\sqrt{b^2+(d-x)^2}} = 0$$

$$n_1\frac{x}{\sqrt{a^2+x^2}} = n_2\frac{d-x}{\sqrt{b^2+(d-x)^2}}$$

$$n_1 sini = n_2 sinr$$

where $sini = \frac{x}{\sqrt{a^2+x^2}}$ and $sinr = \frac{d-x}{\sqrt{b^2+(d-x)^2}}$

$$\frac{sini}{sinr} = \frac{n_2}{n_1}$$

which is law of refraction.

Q. Establish laws of reflection at a spherical surface from Fermat's principle.

Let MM' be a spherical surface of radius $OC = CA = R$. PA be that incident ray and AQ be the reflected ray. i and r be the angle of incidence and reflection. For the triangle APC we write

$$PA = \sqrt{CP^2 + OC^2 + 2.CP.OC.cos\theta}$$

$$PA = \sqrt{CP^2 + R^2 + 2.CP.R.cos\theta} \tag{1}$$

For the triangle AQC we write

$$PA = \sqrt{CQ^2 + OC^2 - 2.CQ.OC.cos\theta}$$

$$PA = \sqrt{CP^2 + R^2 - 2.CQ.R.cos\theta} \tag{2}$$

Here, the optical path is

$$L_{op} = PA + AQ$$

According to Fermat's principle

$$\frac{dL_{op}}{d\theta} = 0$$

$$\frac{d}{d\theta}(PA + AQ) = 0 \tag{3}$$

Differentiating (1) and (2), we get

$$\frac{d(PA)}{d\theta} = -\frac{R.CP}{PA}sin\theta$$

$$\frac{d(AQ)}{d\theta} = \frac{R.CQ}{AQ}sin\theta$$

So, we get from (3)

$$-\frac{R.CP}{PA}sin\theta + \frac{R.CQ}{AQ}sin\theta = 0$$

$$-\frac{CP}{PA} + \frac{CQ}{AQ} = 0$$

$$-\frac{sini}{sin(180^0 - \theta)} + \frac{sinr}{sin\theta} = 0$$

$$-\frac{sini}{sin\theta} + \frac{sinr}{sin\theta} = 0$$

$$sini = sinr$$

$$i = r$$

i.e. angle of incidence=angle of reflection

Q. Derive the formula $\frac{1}{v} + \frac{1}{u} = \frac{2}{R}$ from Fermat's principle.

Let MM' be a spherical surface of radius $OC = CA = R$. PA be that incident ray and AQ be the reflected ray. i and r be the angle of incidence and reflection. For the triangle APC we write

$$PA = \sqrt{CP^2 + OC^2 + 2.CP.OC.cos\theta}$$

$$PA = \sqrt{CP^2 + R^2 + 2.CP.R.cos\theta} \tag{1}$$

Under paraxial approximation the angle θ is small. Hence,

$$cos\theta = 1 - \frac{\theta^2}{v2}$$

Let us assume OP=object distance $= u$. So, we get

$$PA = \left(R^2 + (u-R)^2 + 2R(u-R)\left(1 - \frac{\theta^2}{2}\right)\right)^{1/2}$$

$$= \left(u^2 - R(u-R)\theta^2\right)^{1/2}$$

$$= u\left(1 - \frac{1}{2}\frac{R}{u^2}(u-R)\theta^2\right)$$

$$= u - \frac{1}{2}R^2\left(\frac{1}{R} - \frac{1}{u}\right)\theta^2$$

similarly, for the triangle AQC we write

$$AQ = \sqrt{CQ^2 + OC^2 - 2.CQ.OC.cos\theta}$$

$$AQ = \sqrt{CP^2 + R^2 - 2.CQ.R.cos\theta} \tag{2}$$

Let us assume OQ=object distance $= v$. So, we get

$$AQ = \left(R^2 + (v-R)^2 - 2R(R-v)\left(1 - \frac{\theta^2}{2}\right)\right)^{1/2}$$

$$= \left(v^2 - R(R-v)\theta^2\right)^{1/2}$$

$$= v\left(1 - \frac{1}{2}\frac{R}{v^2}(R-v)\theta^2\right)$$

$$= v - \frac{1}{2}R^2\left(\frac{1}{R} - \frac{1}{v}\right)\theta^2$$

Here, the optical path is

$$L_{op} = PA + AQ$$

We have

$$\frac{dL_{op}}{d\theta} = 0$$

$$\frac{d}{d\theta}(PA + AQ) = 0$$

$$\frac{d}{d\theta}\left(u + v - \frac{1}{2}R^2\left(\frac{2}{R} - \frac{1}{u} - \frac{1}{v}\right)\theta^2\right)$$

$$\frac{2}{R} - \frac{1}{u} - \frac{1}{v} = 0$$

$$\frac{2}{R} = \frac{1}{u} + \frac{1}{v}$$

For this case, $u = -ve, v = -ve, R = -ve$, so we get

$$\frac{2}{-R} = \frac{1}{-u} + \frac{1}{-v})$$

$$\frac{2}{R} = \frac{1}{u} + \frac{1}{v}$$

Q. Establish laws of refraction at a spherical surface from Fermat's principle.

Let MM' be a spherical surface of radius $OC = CA = R$. PA be that incident ray and AQ be the refracted ray. i and r be the angle of incidence and refraction. Let n_1 and n_2 be the refractive index for the two media. For the triangle APC we write

$$PA = \sqrt{CP^2 + OC^2 - 2.CP.OC.cos\theta}$$

$$PA = \sqrt{CP^2 + R^2 - 2.CP.R.cos\theta} \tag{1}$$

For the triangle AQC we write

$$PA = \sqrt{CQ^2 + OC^2 + 2.CQ.OC.cos\theta}$$

$$PA = \sqrt{CP^2 + R^2 + 2.CQ.R.cos\theta} \tag{2}$$

Here, the optical path is

$$L_{op} = n_1PA + n_2AQ$$

According to Fermat's principle

$$\frac{dL_{op}}{d\theta} = 0$$

$$\frac{d}{d\theta}(n_1PA + n_2AQ) = 0 \tag{3}$$

Differentiating (1) and (2), we get

$$\frac{d(PA)}{d\theta} = +\frac{R.CP}{PA}sin\theta$$

$$\frac{d(AQ)}{d\theta} = -\frac{R.CQ}{AQ}sin\theta$$

So, we get from (3)

$$\frac{n_1R.CP}{PA}sin\theta - \frac{n_2R.CQ}{AQ}sin\theta = 0$$

$$\frac{n_1CP}{PA} - \frac{n_2CQ}{AQ} = 0$$

$$\frac{n_1sin(180^0 - i)}{sin\theta} - \frac{sinr}{sin(180^0 - \theta)} = 0$$

$$n_1sini = n_2sinr$$

$$\frac{sini}{sinr} = \frac{n_2}{n_1}$$

which is Snell's law of refraction.

Q. Derive the formula $\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2-n_1}{R}$ from Fermat's principle.

Let MM' be a spherical surface of radius $OC = CA = R$. PA be that incident ray and AQ be the reflected ray. i and r be the angle of incidence and reflection. For the triangle APC we write

$$PA = \sqrt{CP^2 + OC^2 - 2.CP.OC.cos\theta}$$

$$PA = \sqrt{CP^2 + R^2 - 2.CP.R.cos\theta} \tag{1}$$

Under paraxial approximation the angle θ is small. Hence,

$$cos\theta = 1 - \frac{\theta^2}{v2}$$

Let us assume OP=object distance $= u$. So, we get

$$PA = \left(R^2 + (u+R)^2 + 2R(u+R)\left(1 - \frac{\theta^2}{2}\right)\right)^{1/2}$$

$$= \left(u^2 + R(u+R)\theta^2\right)^{1/2}$$

$$= u\left(1 + \frac{1}{2}\frac{R}{u^2}(u+R)\theta^2\right)$$

$$= u + \frac{1}{2}R^2\left(\frac{1}{R} + \frac{1}{u}\right)\theta^2$$

similarly, for the triangle AQC we write

$$AQ = \sqrt{CQ^2 + OC^2 + 2.CQ.OC.cos\theta}$$

$$AQ = \sqrt{CP^2 + R^2 + 2.CQ.R.cos\theta} \qquad (2)$$

Let us assume OQ=object distance $= v$. So, we get

$$AQ = \left(R^2 + (v-R)^2 - 2R(v-R)\left(1 - \frac{\theta^2}{2}\right)\right)^{1/2}$$

$$= \left(v^2 + R(v-R)\theta^2\right)^{1/2}$$

$$= v\left(1 + \frac{1}{2}\frac{R}{v^2}(v-R)\theta^2\right)$$

$$= v + \frac{1}{2}R^2\left(\frac{1}{v} - \frac{1}{R}\right)\theta^2$$

Here, the optical path is

$$L_{op} = n_1 PA + n_2 AQ$$

We have

$$\frac{dL_{op}}{d\theta} = 0$$

$$\frac{d}{d\theta}(n_1 PA + n_2 AQ) = 0$$

$$\frac{d}{d\theta}\left(u + v - \frac{1}{2}R^2\left(-\frac{n_2 - n_1}{R} + \frac{n_1}{u} + \frac{n_2}{v}\right)\theta^2\right)$$

$$-\frac{n_2 - n_1}{R} + \frac{n_1}{u} + \frac{n_2}{v} = 0$$

For this case, $u = -ve, v = -ve, R = -ve$, so we get

$$-\frac{n_2 - n_1}{R} + \frac{n_1}{-u} + \frac{n_2}{v} = 0$$

$$\frac{n_2 - n_1}{R} = \frac{n_2}{v} - \frac{n_1}{u}$$

Q. Using Fermat's principle obtain the reflection formula for the virtual image formation in a convex mirror.

Let MM' be a spherical surface of radius $OC = CA = R$. PA be that incident ray and AQ be the reflected ray. i and r be the angle of incidence and reflection. For the triangle APC we write

$$PA = \sqrt{CP^2 + AC^2 - 2.CP.AC.cos\theta}$$

$$PA = \sqrt{CP^2 + R^2 - 2.CP.R.cos\theta} \tag{1}$$

Under paraxial approximation the angle θ is small. Hence,

$$cos\theta = 1 - \frac{\theta^2}{v2}$$

Let us assume OP=object distance $= u$. So, we get

$$PA = \left(R^2 + (u+R)^2 + 2R(u+R)\left(1 - \frac{\theta^2}{2}\right)\right)^{1/2}$$

$$= \left(u^2 + R(u+R)\theta^2\right)^{1/2}$$

$$= u\left(1 + \frac{1}{2}\frac{R}{u^2}(u+R)\theta^2\right)$$

$$= u + \frac{1}{2}R^2\left(\frac{1}{R} + \frac{1}{u}\right)\theta^2$$

similarly, for the triangle AQC we write assuming OQ=object distance $= v$. So, we get

$$AQ = v + \frac{1}{2}R^2\left(\frac{1}{v} - \frac{1}{R}\right)\theta^2$$

Here, the optical path is

$$L_{op} = PA - AQ$$

Here the minus sign is taken because the rays at A point away from Q. We have

$$\frac{dL_{op}}{d\theta} = 0$$

$$\frac{d}{d\theta}(PA - AQ) = 0$$

$$\frac{d}{d\theta}\left(u + \frac{1}{2}R^2\left(\frac{1}{R} + \frac{1}{u}\right)\theta^2 - v + \frac{1}{2}R^2\left(\frac{1}{v} - \frac{1}{R}\right)\theta^2\right) = 0$$

As $\theta \neq= 0$,

$$\frac{1}{u} - \frac{1}{v} = -\frac{2}{R}$$

Again, $u = -Ve, v = +ve, R = +ve$. So we get

$$\frac{1}{-u} - \frac{1}{+v} = -\frac{2}{+R}$$

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{R}$$